Exact energy eigenvalues of the generalized Dirac-Coulomb equation via a modified similarity transformation

Omar Mustafa
Department of Physics, Eastern Mediterranean University
G. Magusa, North Cyprus, Mersin 10 - Turkey
Thabit Barakat
Department of Civil Engineering, Near East University
Lefkoşa, North Cyprus, Mersin 10 - Turkey

Abstract

With the aid of a modified similarity transformation we have obtained exact energy eigenvalues of the generalized Dirac - Coulomb equation. This equation consists of the time component of the Lorentz 4-vector potential $V_v(r) = -A_1/r$, and a Lorentz scalar potential $V_s(r) = -A_2/r$. The transformed radial equations are so simple so that their solutions are inferred from the conventional solutions of the Schrödinger - Coulomb equation.

I Introduction

Recently, Su [1] has used a simple similarity transformation to bring the radial wave equation of Dirac - Coulomb problem into a form nearly identical to those of the Schrödinger and Klein - Gordon equations. With the aid of the confluent hypergeometric functions, he was able to come out with the exact Sommerfeld - Dirac discrete spectrum. Wong [2] has followed the same procedure to obtain the exact solution of the N - dimensional Dirac - Coulomb equation. The similarity transformation S they have used was obtained by Biedenharn [3] and Wong and Yeh [4].

However, a scalar interaction is of great importance in the context of the relativistic quark model. It is employed for describing the magnetic moment [5] and avoiding the Klein paradox risen from the quarkonium confining potentials [6,7]. Therefore, the search for exact solutions to problems concerned with the scalar interaction is of special significance. Tutik [8] has obtained an exact solution to the bound states problem for the N - dimensional generalized Dirac - Coulomb equation, whose potential contains both the Lorentz 4 - vector and Lorentz scalar terms of Coulomb form. Tutik has considered only the positive energy solution, i.e. the particle case.

In this paper we modify the similarity transformation used by Su [1] and Wong [2] to simplify the generalized Dirac - Coulomb equation and bring it into a form almost identical to those of the Schrödinger and Klein - Gordon equations in a Coulomb field. The constants involved in the transformation matrix S are determined in such a way that the resulting second - order radial differential equations do not include fist - derivatives of the 4-vector and/or of the scalar potential. One can thus put the transformed radial equations in suggestive forms for which their solutions can be inferred from the known nonrelativistic solution of the Coulomb problem.

In Sec II the generalized Dirac - Coulomb equation is transformed under a modified similarity transformation. Comparing the transformed radial equations with the Schrödinger - Coulomb problem we extract the exact energy eigenvalues of the generalized Dirac - Coulomb equation. In the same section we discuss various special cases concerning the exact energy eigenvalues. We conclude in Sec III.

II A modified similarity transformation for the generalized Dirac-Coulomb equation.

The generalized Dirac-Coulomb equation involves a Coulomb potential in the form of a superposition of the Lorentz - vector and Lorentz - scalar terms $V_v(r) = -A_1/r$ and $V_s(r) = -A_2/r$, respectively. The Dirac equation thus reads (with the units $\hbar = c = 1$)

$$H\Psi = E\Psi, \tag{1}$$

where

$$H = \vec{\alpha} \cdot \vec{p} + \beta(m - A_2/r) - A_1/r, \tag{2}$$

and the Dirac matrices $\vec{\alpha}$ and β have their usual meanings. Applying a similarity transformation [1, 2] to the Dirac equation one gets

$$H^{'}\Psi^{'} = E\Psi^{'}, \tag{3}$$

with

$$H^{'} = SHS^{-1}, \tag{4}$$

$$\Psi^{'} = S\Psi, \tag{5}$$

and

$$S = a + ib\beta\vec{\alpha} \cdot \hat{r}, \tag{6}$$

where $\hat{r}$ is the unit vector $\vec{r}/r$ and a and b are real constants to be determined. For the central potentials above, the transformed wave function is given by

$$\Psi^{'} = \left[\begin{array}{c} iR(r)\Phi^{l}_{jm} \\ Q(r)\vec{\sigma} \cdot \hat{r}\Phi^{l}_{jm} \end{array} \right]. \tag{7}$$

In a straightforward manner one can calculate

$$E\Psi^{'} = SHS^{-1}\Psi^{'}, \tag{8}$$

to obtain two coupled equations for $R(r)$ (the upper component) and $Q(r)$ (the lower component);

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} R(r) \\ Q(r) \end{bmatrix} = E \begin{bmatrix} R(r) \\ Q(r) \end{bmatrix}, \tag{9}$$

with

$$M_{11} = (m - A_2/r)cosh\theta + (\partial_r + 1/r)sinh\theta - A_1/r, \tag{10}$$

$$M_{12} = -[(m - A_2/r)sinh\theta + (\partial_r + 1/r)cosh\theta - K/r], \tag{11}$$

$$M_{21} = (m - A_2/r)sinh\theta + (\partial_r + 1/r)cosh\theta + K/r, \tag{12}$$

$$M_{22} = -[(m - A_2/r)cosh\theta + (\partial_r + 1/r)sinh\theta + A_1/r], \tag{13}$$

where $K = \tilde{\omega}(j + 1/2)$, $\tilde{\omega} = \mp 1$ for $l = j \mp 1/2$, $cosh\theta = (a^2 + b^2)/(a^2 - b^2)$, and $sinh\theta = 2ab/(a^2 - b^2)$. The coupled equations for $R(r)$ and $Q(r)$ are

$$M_{11}R(r) + M_{12}Q(r) = ER(r), \tag{14}$$

and

$$M_{21}R(r) + M_{22}Q(r) = EQ(r). \tag{15}$$

Multiply Eq(14) by $sinh\theta$, Eq(15) by $cosh\theta$ and subtract to obtain

$$Q(r) = \frac{1}{\xi_1}[\partial_r + \frac{1}{r} + \frac{K}{r}cosh\theta + \frac{A_1}{r}sinh\theta + Esinh\theta]R(r), \tag{16}$$

Next, multiply Eq(14) by $cosh\theta$, Eq(15) by $sinh\theta$ and subtract to get

$$R(r) = \frac{1}{\xi_2}[\partial_r + \frac{1}{r} - \frac{K}{r}cosh\theta - \frac{A_1}{r}sinh\theta - Esinh\theta]Q(r), \tag{17}$$

where

$$\xi_1 = m - \frac{A_2}{r} + \frac{A_1}{r}cosh\theta + \frac{K}{r}sinh\theta + Ecosh\theta, \tag{18}$$

and

$$\xi_2 = m - \frac{A_2}{r} - \frac{A_1}{r} cosh\theta - \frac{K}{r} sinh\theta - E cosh\theta. \tag{19}$$

Incorporating the regular asymptotic behaviour of the radial functions near the origin; i.e. $R(r) \sim a_1 r^{\gamma-1}$ and $Q(r) \sim a_2 r^{\gamma-1}$ at $r \sim 0$, where the constant terms proportional to mass and energy can be neglected, one gets

$$\gamma = \sqrt{K^2 - A_1^2 + A_2^2}. \tag{20}$$

The positive sign of the square root has been chosen to allow not only the normalization of the wave functions, but also the expectation value of each partial operator within the transformed Hamiltonian.

To attain a great simplification in solving the radial equations, Eq(16) and (17), one may choose

$$sinh\theta = [|K| A_2 - \tilde{\omega} A_1 \gamma]/[K^2 - A_1^2], \tag{21}$$

and

$$cosh\theta = [|K| \gamma - \tilde{\omega} A_1 A_2]/[K^2 - A_1^2]. \tag{22}$$

In fact one gets

$$\xi_1 = m - A_2/r + \tilde{\omega} A_2/r + E cosh\theta, \tag{23}$$

and

$$\xi_2 = m - A_2/r - \tilde{\omega} A_2/r - E cosh\theta. \tag{24}$$

Provided that

$$\tilde{\omega} = \left[\begin{array}{llll} +1 & for & K > 0; & n = n_r + |K| + 1 \\ -1 & for & K < 0; & n = n_r + |K| \end{array} \right. \tag{25}$$

where n_r and n are the radial and principle quantum numbers, respectively.

Moreover, for $\tilde{\omega} = +1$, Eq(23) and (24) read

$$\xi_1 = m + E cosh\theta, \tag{26}$$

and

$$\xi_2 = m - 2A_2/r + E cosh\theta, \tag{27}$$

which when substituted in (16) and (17), eliminating $Q(r)$, imply

$$[E^2 - m^2]R(r) = \left[-\partial_r^2 - \frac{2}{r}\partial_r + \frac{(\gamma^2+\gamma)}{r^2} - \frac{2(mA_2 + A_1E)}{r}\right]R(r). \quad (28)$$

For $\tilde{\omega} = -1$, Eq(23) and (24) read

$$\xi_1 = m - 2A_2/r + Ecosh\theta, \quad (29)$$

and

$$\xi_2 = m - Ecosh\theta, \quad (30)$$

which in turn when substituted in (16) and (17), eliminating $R(r)$, imply

$$[E^2 - m^2]Q(r) = \left[-\partial_r^2 - \frac{2}{r}\partial_r + \frac{(\gamma^2+\gamma)}{r^2} - \frac{2(mA_2 + A_1E)}{r}\right]Q(r). \quad (31)$$

The first derivatives can be removed by defining $R(r) = r^{-1}\phi(r)$ and $Q(r) = r^{-1}q(r)$ to obtain

$$[E^2 - m^2]\phi(r) = \left[-\partial_r^2 + \frac{(\gamma^2+\gamma)}{r^2} - \frac{2(mA_2 + A_1E)}{r}\right]\phi(r) \quad (32)$$

for $\tilde{\omega} = +1$, and

$$[E^2 - m^2]q(r) = \left[-\partial_r^2 + \frac{(\gamma^2+\gamma)}{r^2} - \frac{2(mA_2 + A_1E)}{r}\right]q(r) \quad (33)$$

for $\tilde{\omega} = -1$.

It is obvious that Eqs(32) and (33) are in a form nearly identical to that of the corresponding radial wave equation of Schrödinger in the Coulomb field [1-4, 8-11]. Their solutions can thus be inferred from the known Schrödinger - Coulomb solution [9-11]. Moreover, the choices of *sinhθ* and *coshθ*, consequently the S transformation, have successfully transformed the coupled radial equations of the generalized Dirac - Coulomb equation into simple forms which are exactly soluble.

Inferred from the known nonrelativistic solution of the Coulomb problem the exact energy eigenvalues for Eqs(32) and (33) are given through the relation

$$E^2 - m^2 = -[mA_2 + A_1E]^2/\tilde{n}^2, \quad (34)$$

which implies

$$\frac{E}{m} = -\frac{A_1 A_2}{\tilde{n}^2 + A_1^2} \pm \left[\left(\frac{A_1 A_2}{\tilde{n}^2 + A_1^2} \right)^2 + \frac{(\tilde{n}^2 - A_2^2)}{\tilde{n}^2 + A_1^2} \right]^{1/2}. \tag{35}$$

Provided $\tilde{n} = n_r + \gamma + 1$, where n_r and γ are defined through Eqs(25) and (20), respectively. Eq(35) represents the exact energy eigenvalues of the Dirac equation with an attractive central potential that contains both the time component of a Lorentz 4-vector term, $V_v(r) = -A_1/r$, and the Lorentz - scalar term, $V_s(r) = -A_2/r$ [9]. It should be noted that in the case of the superposition of the central potentials above K^2 is required to be larger than $A_1^2 - A_2^2$ otherwise γ becomes imaginary, causing the breakdown of the bound state solution. Moreover, this result agrees with that obtained by Tutik [8] when only the positive energy (the particle case) solution is considered.

In the context of Eq(35) various special cases deserve attention.

For $A_1 = 0$ and $A_2 \neq 0$, $\gamma = \sqrt{K^2 + A_2^2}$, and

$$E = \pm \left[1 - \frac{A_2^2}{\tilde{n}^2} \right]^{1/2}. \tag{36}$$

Which is the exact eigenvalue spectrum of the three - dimensional Dirac equation with the scalar coupling potential $V_s(r) = -A_2/r$ [9].

For $A_2 = 0$ and $A_1 \neq 0$, $\gamma = \sqrt{K^2 - A_1^2}$,

$$\tilde{n} = \frac{A_1 E}{[m^2 - E^2]^{1/2}} > 0, \tag{37}$$

and

$$E = m \left[1 + \frac{A_1^2}{\tilde{n}^2} \right]^{-1/2}. \tag{38}$$

Where the negative square root is not possible for it yields a contradiction to Eq(37). Obviously this result is identical with the known Sommerfeld's fine - structure formula [8,9].

For $A_1 = A_2 = A$, $\gamma = |K|$ and

$$E = m \left[\frac{-A^2}{\tilde{n}^2 + A^2} \pm \frac{\tilde{n}^2}{\tilde{n}^2 + A^2} \right]. \tag{39}$$

The negative sign yields a solution $E = -m$ which is invalid for it contradicts Eq(34). For the positive sign, it follows that

$$E = m \left[1 - \frac{2A^2}{\tilde{n}^2 + A^2} \right]. \tag{40}$$

This result allows positive and negative energy solutions. However, Eq(34) suggests that positive and negative energy solutions are possible if and only if $\tilde{n}^2 > A^2$ and $\tilde{n}^2 < A^2$, respectively, otherwise impossible for they contradict Eq(34). Moreover, it is clear from Eq(40) that at the limit $A \rightarrow \infty$ the energy E approaches $-m$ asymptotically and contradicts Eq(34).

III Conclusions and remarks

In this paper we have modified the similarity transformation S used by Su [1] and Wong [2] to obtain exact energy eigenvalues of the generalized Dirac - Coulomb equation. The modified transformation reduces to that used by Su [1] and Wong [2] when the scalar potential $V_s(r)$ is switched off.

To the best of our knowledge, this is the first time that a similarity transformation is used to obtain the exact energy eigenvalues of the generalized Dirac - Coulomb equation. The importance of this new representation of the transformation S, as suggested by Eqs (28) and (31), lies in the fact that it enables us to treat the generalized Dirac - Coulomb radial functions $R(r)$ and $Q(r)$ as the precise analogs to the radial wave functions of the nonrelativistic Coulomb problem. Provided, as evident from Eqs(28) and (31), that the integer orbital angular momentum of the radial Schrödinger - Coulomb equation becomes in the relativistic Coulomb problem a noninteger (irrational) orbital angular momentum; i.e. $l_{Schr\ddot{o}dinger} \equiv \gamma_{Dirac-Coulomb}$.

References

[1] J Y Su, Phys Rev **A32**, 3251 (1985).

[2] M K F Wong, J Math Phys **31**, 1677 (1990).

[3] L C Biedenharn, Phys Rev **126**, 845 (1962).

[4] M K F Wong and H Y Yeh, Phys Rev **D25**, 3396 (1982).

[5] H J Lipkin and A Tavkhelidze, Phys Lett **17**, 331 (1965).

[6] C L Critchfield, Phys Rev **D12**, 923 (1975).

[7] J F Gunion and L F Li, Phys Rev **D12**, 3583 (1975).

[8] R S Tutik, J Phys **A**; Math Gen **25**, L413 (1992).

[9] W Greiner, *Relativistic Quantum Mechanics* (Springer - Verlag Berlin Heidelberg, 1990).

[10] M D Scadron, *Advanced Quantum Theory* (Springer, New York, 1979).

[11] L I Schiff, *Quantum Mechanics*, 3rd ed (McGrow - Hill, New York, 1968).